Chapter 10

REGENERATION

1 Introduction

In this last chapter we finally focus on the third topic of the book, regeneration. Regenerative processes are generalizations of Markov chains and renewal processes, which we considered in Chapter 2. We shall look at several kinds of regeneration, and as in Chapter 2, the aspects we concentrate on are coupling, stationarity (and its generalizations), and total variation asymptotics.

In Section 2 we establish notation and then consider briefly the one-sided counterpart of the two-sided stationarity theory of Chapter 8.

In Section 3 we consider *classical* regeneration. A stochastic process is regenerative in the classical sense if there are random times where it starts anew independently of the past, like a recurrent Markov chain at the times of visits to a fixed reference state. The regeneration times form a renewal process and split the stochastic process into a sequence of cycles that are i.i.d. and independent of a possible initial delay.

In Section 4 we consider *wide-sense* regeneration. Wide-sense regeneration allows the future after regeneration to depend on the past as long as the future is independent of the past regeneration times. This is the type of regeneration occurring in so-called Harris chains, but for a simple example consider a recurrent Markov chain and let $l > 0$ be fixed: l time units after visiting a fixed reference state the chain regenerates in the wide sense (but typically not in the classical sense). The regeneration times still form a renewal process, but the cycles are only stationary and need not be independent.

In Section 5 we move on to consider *time-inhomogeneous* regeneration. Time-inhomogeneous regeneration allows the future after regeneration to depend on the time of regeneration. This is the type of regeneration occurring in time-inhomogeneous Markov chains with a recurrent state: if such a Markov chain visits this state at time t, then the future after time t is independent of the past before time t but has a distribution that depends on t. In this case the sequence of cycles need no longer be stationary and the regeneration times need not form a renewal process, they only form an increasing discrete-time Markov process (note that a renewal process is an example of an increasing discrete-time Markov process).

Section 6 contains a coupling construction for time-inhomogeneous regenerative processes. This construction is an elaboration on the classical coupling (Chapter 2). In Section 7 we investigate the coupling time thoroughly to obtain results on uniform convergence and rates of convergence.

In Section 8 we introduce asymptotics *from-the-past*. Ordinary asymptotics are *to-the-future*: we start a process at time zero and observe it in the far future to obtain a stationary limit process. Asymptotics from-the-past is the reversal of this procedure: we start a process in the remote past and observe it from any fixed time t onwards to obtain a (typically) *nonstationary* limit process. In the time-inhomogeneous case we cannot expect to obtain a limit process by going to-the-future (unless the time-inhomogeneity disappears asymptotically), but coming in from-the-past turns out to yield a limit process, a nonstationary one because of the time-inhomogeneity.

In Section 9 we consider *taboo* regeneration. Taboo regeneration means basically that the process regenerates in the classical sense while not entering a fixed region of the state space (taboo region), like a transient Markov chain while it stays in a finite irreducible set of states. In this case the regeneration times form a possibly terminating renewal process. We shall consider the existence of *taboo limits*: we start a process at time zero and observe it in the far future, conditionally on not yet having entered the taboo region, to obtain a limit process. A taboo regenerative process becomes a time-inhomogenous regenerative process under this conditioning, and thus the limit theory from the time-inhomogeneous case applies.

In Section 10 we consider *taboo stationarity*, the characterizing property of a taboo limit process, and work out the structure of the taboo limit process in the taboo regenerative case. This structure is quite different from, but analogous to, the structure of the stationary version of a cycle-stationary process (Chapter 8).

Section 11 rounds off with *coupling from-the-past*, a perfect simulation method for generating observations from the stationary, nonstationary, and taboo stationary (quasi-stationary) limits of finite state space Markov chains.

As the above description indicates, many themes from the previous chapters converge in the next two sections, to be further developed and extended in the remaining sections.