Rate and coupling results follow easily from Theorem 8.5 using the distribution functions $\bar{G}$ and $\bar{F}$ introduced in the proof of Theorem 9.4. We obtain the moment conditions for $\bar{G}$ and $\bar{F}$ by placing moment conditions on the ingredients in their definition.

For the uniformity results the common density and mass function part is easy. However, some care must be taken with the constants a and b in Theorem 9.3. They must be traced through the proof of Theorem 9.2 (using the uniform convergence in Theorem 3.3 rather than using Theorems 10.1 and 10.2 from Chapter 2).

The reader may also have noted that we left out the statements on existence of a two-sided limit process. We did this because the issue will be considered in great detail in the next section, where we shall establish the explicit structure of the two-sided limit process.

10 Taboo Stationarity

In this section we shall consider the taboo counterpart of stationarity. Stationarity means that the distribution of a process does not change by nonrandom time shifts. This is the characterizing property of any two-sided limit process obtained by shifting the time origin of a one-sided process to the far future. Similarly, *taboo stationarity* means that the distribution of a process does not change by nonrandom time shifts *under taboo*. This is the characterizing property of any two-sided limit process obtained by shifting the origin of a one-sided process to the far future under taboo up to the new time origin.

We begin by defining taboo stationarity for general two-sided processes and show that it is the characterizing property of a taboo limit (Theorem 10.1). We then establish a basic but amazingly simple structural characterization of taboo stationary processes (independent-exponential-shift-to-the-past, Theorem 10.2).

After this we return to taboo regeneration and explicitly construct a taboo stationary version of a taboo regenerative process (Theorems 10.3 through 10.6). This is the taboo counterpart of the construction of a stationary process in Chapter 8. We finally show (Theorem 10.7) that this taboo stationary version is indeed the limit process in Theorem 9.4 above.

10.1 Taboo Stationary Stochastic Processes – Definition

Consider a pair (Z^*, Γ^*), where Γ^* is a nonnegative finite random time and

$$Z^* = (Z_s^*)_{s\in\mathbb{R}}$$

is a two-sided shift-measurable stochastic process. Recall that θ_t in this chapter (see Section 2 and Section 9.1 for formal details) denotes the one-sided shift-map:

$$\theta_t Z^* = (Z^*_{t+s})_{s\in[0,\infty)}, \quad t \in \mathbb{R}, \qquad \text{(one-sided shift)}.$$

Let $\overset{\leftrightarrow}{\theta}_t$ denote the two-sided shift-map:

$$\overset{\leftrightarrow}{\theta}_t Z^* = (Z^*_{t+s})_{s\in\mathbb{R}}, \quad t \in \mathbb{R}, \qquad \text{(two-sided shift)}.$$

Call Z^* *taboo stationary* with *taboo time* Γ^* if shift under taboo does not change the distribution of the pair (Z^*, Γ^*), that is, if

$$\mathbf{P}((\overset{\leftrightarrow}{\theta}_t Z^*, \Gamma^* - t) \in \cdot|\Gamma^* > t) = \mathbf{P}((Z^*, \Gamma^*) \in \cdot), \quad t \in [0, \infty). \quad (10.1)$$

Call (Z^*, Γ^*) *taboo stationary* if this holds. Note that if we shift the origin back, then (10.1) yields $\mathbf{P}((Z^*, \Gamma^*) \in \cdot|\Gamma^* > t) = \mathbf{P}((\overset{\leftrightarrow}{\theta}_{-t} Z^*, \Gamma^* + t) \in \cdot)$ for $t \in [0, \infty)$. Thus (10.1) is equivalent to the following condition:

$$\mathbf{P}((\overset{\leftrightarrow}{\theta}_t Z^*, \Gamma^* - t) \in \cdot|\Gamma^* > t) = \mathbf{P}((Z^*, \Gamma^*) \in \cdot|\Gamma^* > -t), \quad t \in \mathbb{R}.$$

We shall now show that taboo stationarity is the characterizing property of a total variation taboo limit.

Theorem 10.1. *A pair (Z^*, Γ^*) is taboo stationary if and only if there is a pair (Z, Γ), where $Z = (Z_s)_{s\in[0,\infty)}$ is a one-sided shift-measurable process and Γ is a nonnegative finite random time, such that*

$$\mathbf{P}((\theta_{t-h} Z, \Gamma - t) \in \cdot|\Gamma > t) \xrightarrow{tv} \mathbf{P}((\theta_{-h} Z^*, \Gamma^*) \in \cdot), \quad t \to \infty, \quad (10.2)$$

for all $h \in [0, \infty)$.

Proof. If (10.1) holds, then so does (10.2) with $(Z, \Gamma) := (\theta_0 Z^*, \Gamma^*)$. In order to establish the converse [that (10.2) implies (10.1)], assume that (10.2) holds. Take $x \in [0, \infty)$ and $h \in [x, \infty)$ and note that (10.2) implies [with h replaced by $h - x$] that

$$\begin{aligned}&\mathbf{P}((\theta_{t-(h-x)} Z, \Gamma - t - x) \in \cdot, \Gamma - t > x|\Gamma > t)\\ &\quad \xrightarrow{tv} \mathbf{P}((\theta_{-(h-x)} Z^*, \Gamma^* - x) \in \cdot, \Gamma^* > x), \quad t \to \infty.\end{aligned}$$

Divide by $\mathbf{P}(\Gamma - t > x|\Gamma > t)$ on the left and by the limit $\mathbf{P}(\Gamma^* > x)$ on the right [and note that $\theta_{t-(h-x)} Z = \theta_{(t+x)-h} Z$ and $\theta_{-(h-x)} Z^* = \theta_x \theta_{-h} Z^*$] to obtain that as $t \to \infty$,

$$\begin{aligned}&\mathbf{P}((\theta_{(t+x)-h} Z, \Gamma - t - x) \in \cdot|\Gamma > t + x)\\ &\quad \xrightarrow{tv} \mathbf{P}((\theta_x \theta_{-h} Z^*, \Gamma^* - x) \in \cdot|\Gamma^* > x).\end{aligned}$$

According to (10.2), the left-hand side tends also to $\mathbf{P}((\theta_{-h} Z^*, \Gamma^*) \in \cdot)$. Since the two limits must be identical, we have [replace x by t] that

$$\mathbf{P}((\theta_t \theta_{-h} Z^*, \Gamma^*) \in \cdot|\Gamma^* > t) = \mathbf{P}((\theta_{-h} Z^*, \Gamma^*) \in \cdot), \quad 0 \leqslant t \leqslant h.$$

Since h is arbitrary, this yields (10.1). $\square$

10.2 Basic Structural Characterization

According to the following theorem, taboo stationarity is characterized by an independent-exponential-shift-to-the-past. That is, if Z' is some two-sided shift-measurable process and V is exponential and independent of Z', then

$$(\overleftrightarrow{\theta}_{-V} Z', V) \text{ is always taboo stationary;}$$

and conversely, all taboo stationary processes are of this form.

Theorem 10.2. *The pair $(Z^*, \varGamma^*)$ is taboo stationary if and only if $\varGamma^*$ is exponential and independent of $\overleftrightarrow{\theta}_{\varGamma^*} Z^*$.*

PROOF. Suppose $(Z^*, \varGamma^*)$ is taboo stationary. From (10.1) we obtain

$$\mathbf{P}(\varGamma^* - t \in \cdot | \varGamma^* > t) = \mathbf{P}(\varGamma^* \in \cdot), \quad t \in [0, \infty),$$

which is the standard characterization of exponentiality. Moreover,

$$\overleftrightarrow{\theta}_{\varGamma^*} Z^* = \overleftrightarrow{\theta}_{\varGamma^* - t} \overleftrightarrow{\theta}_t Z^*, \quad t \in [0, \infty),$$

that is, $\overleftrightarrow{\theta}_{\varGamma^*} Z^*$ is the same measurable mapping of $(\overleftrightarrow{\theta}_t Z^*, \varGamma^* - t)$ for all $t \in [0, \infty)$. This together with (10.1) yields

$$\mathbf{P}(\overleftrightarrow{\theta}_{\varGamma^*} Z^* \in \cdot | \varGamma^* > t) = \mathbf{P}(\overleftrightarrow{\theta}_{\varGamma^*} Z^* \in \cdot), \quad t \in [0, \infty).$$

Multiply by $\mathbf{P}(\varGamma^* > t)$ to obtain

$$\mathbf{P}(\overleftrightarrow{\theta}_{\varGamma^*} Z^* \in \cdot, \varGamma^* > t) = \mathbf{P}(\overleftrightarrow{\theta}_{\varGamma^*} Z^* \in \cdot)\mathbf{P}(\varGamma^* > t), \quad t \in [0, \infty),$$

that is, $\overleftrightarrow{\theta}_{\varGamma^*} Z^*$ and $\varGamma^*$ are independent.

Conversely, suppose $\varGamma^*$ is exponential and independent of $\overleftrightarrow{\theta}_{\varGamma^*} Z^*$. Since $\varGamma^*$ is exponential, we have, for all z in the path set H,

$$\mathbf{P}((\overleftrightarrow{\theta}_{t - \varGamma^*} z, \varGamma^* - t) \in \cdot | \varGamma^* > t) = \mathbf{P}((\overleftrightarrow{\theta}_{-\varGamma^*} z, \varGamma^*) \in \cdot), \quad t \in [0, \infty).$$

Since $\varGamma^*$ and $\overleftrightarrow{\theta}_{\varGamma^*} Z^*$ are independent, we may replace z by $\overleftrightarrow{\theta}_{\varGamma^*} Z^*$ to obtain [since $\overleftrightarrow{\theta}_{t-\varGamma^*} \overleftrightarrow{\theta}_{\varGamma^*} Z^* = \overleftrightarrow{\theta}_t Z^*$ and $\overleftrightarrow{\theta}_{-\varGamma^*} \overleftrightarrow{\theta}_{\varGamma^*} Z^* = Z^*$]

$$\mathbf{P}((\overleftrightarrow{\theta}_t Z^*, \varGamma^* - t) \in \cdot | \varGamma^* > t) = \mathbf{P}((Z^*, \varGamma^*) \in \cdot), \quad t \in [0, \infty),$$

that is, $(Z^*, \varGamma^*)$ is taboo stationary. $\square$

Remark 10.1. In the above we could consider Z^* jointly with a nondecreasing sequence of $(-\infty, \infty]$ valued random times $S^* = (S_k^*)_{-\infty}^{\infty}$ satisfying, for all $n \geqslant 0$,

$$-\infty \leftarrow \cdots < S_{-2}^* < S_{-1}^* < 0 \leqslant S_0^* < \cdots < S_n^* \quad \text{on} \quad \{S_n^* < \infty\}.$$

For $t \in \mathbb{R}$, let $\overset{\leftrightarrow}{\theta}_t$ denote the following two-sided joint shift-maps:

$$\overset{\leftrightarrow}{\theta}_t(Z^*, S^*) = (\overset{\leftrightarrow}{\theta}_t Z^*, (S^*_{N^*_{t-}+k})_{-\infty}^{\infty}) \quad \text{where} \quad N^*_{t-} = \inf\{k : S_k^* \geqslant t\},$$

$$\overset{\leftrightarrow}{\theta}_t(Z^*, S^*, \Gamma^*) = (\overset{\leftrightarrow}{\theta}_t(Z^*, S^*), \Gamma^* - t).$$

The triple (Z^*, S^*, Γ^*) is *taboo stationary* if for $t \in [0, \infty)$,

$$\mathbf{P}(\overset{\leftrightarrow}{\theta}_t(Z^*, S^*, \Gamma^*) \in \cdot | \Gamma^* > t) = \mathbf{P}((Z^*, S^*, \Gamma^*) \in \cdot).$$

Both Theorem 10.1 and Theorem 10.2 hold with Z^* replaced by (Z^*, S^*).

10.3 Back to Taboo Regeneration – Intuitive Motivation

We now return to the topic of the last section, taboo regeneration. The task for the rest of this section is to construct a taboo stationary version (Z^*, S^*, Γ^*) of a taboo regenerative (Z, S, Γ). Here is an attempt at motivating this construction intuitively in the proper taboo regenerative case, that is, not in the wide-sense case but in the case when (Z, S, Γ) satisfies (9.1); see Figure 10.1.

Think of (Z^*, S^*, Γ^*) as a taboo limit of (Z, S, Γ). Then the following guesses seem reasonable. The cycles of (Z^*, S^*) coming in from the past, $\dots, C_{-2}^*, C_{-1}^*$, should be i.i.d. and independent of the cycle C_0^* straddling zero. Moreover, conditionally on $\{S_0^* < \infty\}$, the cycles $\dots, C_{-2}^*, C_{-1}^*, C_0^*$ should be independent of the future $\theta_{S_0^*}(Z^*, S^*, \Gamma^*)$, which should behave as the zero-delayed version of (Z, S, Γ).

In order to have a complete description of (Z^*, S^*, Γ^*) there are still three guesses missing: the distribution of C_{-1}^*, the distribution of C_0^*, and the position of zero in the cycle C_0^*. But we shall not proceed further along this path [for the complete description of (Z^*, S^*, Γ^*), see the comment following Theorem 10.4] because it turns out to be easier to consider another triple $(\tilde{Z}, \tilde{S}, \tilde{\Gamma})$ defined as follows:

$$(\tilde{Z}, \tilde{S}, \tilde{\Gamma}) = \overset{\leftrightarrow}{\theta}_{R^*}(Z^*, S^*, \Gamma^*) \qquad \text{(see Figure 10.1)},$$

where R^* is the last S_k^* in $(-\infty, \Gamma^*]$, that is,

$$R^* := S^*_{N^*_{\Gamma^*}-1} = \sup\{S_k^* : k \in \mathbb{Z} \text{ and } S_k^* \leqslant \Gamma^*\}.$$

In the light of Theorem 10.2, Γ^* should be exponential and independent of $\overset{\leftrightarrow}{\theta}_{\Gamma^*}(Z^*, S^*)$. But $\overset{\leftrightarrow}{\theta}_{\Gamma^*}(Z^*, S^*) = \overset{\leftrightarrow}{\theta}_{\tilde{\Gamma}}(\tilde{Z}, \tilde{S})$, and $-\tilde{\Gamma}$ is the initial point of the $\overset{\leftrightarrow}{\theta}_{\Gamma^*}(Z^*,S^*)$-cycle straddling zero. Thus $(\tilde{Z},\tilde{S},\tilde{\Gamma})$ is a measurable mapping of $\overset{\leftrightarrow}{\theta}_{\Gamma^*}(Z^*,S^*)$ and should therefore be independent of Γ^*. Thus we should obtain (Z^*,S^*) as follows:

$$(Z^*,S^*) = \overset{\leftrightarrow}{\theta}_{-\Gamma^*}\overset{\leftrightarrow}{\theta}_{\tilde{\Gamma}}(\tilde{Z},\tilde{S})$$

where

Γ^* is exponential and independent of $(\tilde{Z},\tilde{S},\tilde{\Gamma})$.

Now let us guess at the structure of $(\tilde{Z},\tilde{S},\tilde{\Gamma})$. We have already guessed that the cycles of (Z^*,S^*) are i.i.d. before time zero. Since (Z^*,S^*) is obtained by shifting the origin of $\overset{\leftrightarrow}{\theta}_{\tilde{\Gamma}}(\tilde{Z},\tilde{S})$ independently to the past, this suggests that the same should hold for $\overset{\leftrightarrow}{\theta}_{\tilde{\Gamma}}(\tilde{Z},\tilde{S})$, that is,

$\ldots,\tilde{C}_{-1},\tilde{C}_0$ should be i.i.d. copies of C^*_{-1}.

The naive guess would be that C^*_{-1} is like the first cycle C_1 of the zero-delayed version, conditioned on the event $\{\Gamma^\circ \geqslant X_1\}$. But this is not the case. So now let us give up guessing and simply state the upcoming results (in the proper taboo regenerative case).

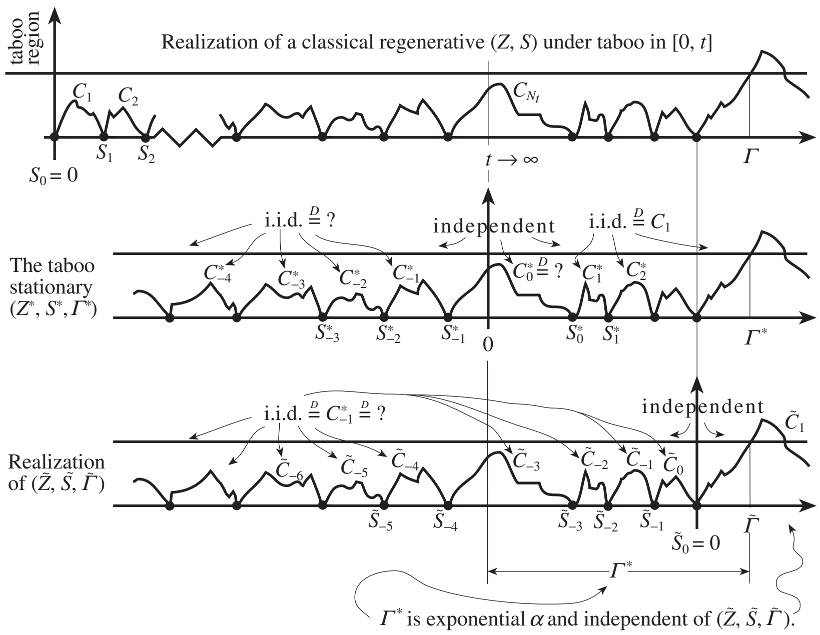

FIGURE 10.1. The structure of the taboo stationary version (Z^*,S^*,Γ^*).

It turns out that an exponential biasing of the cycle-length X_1 (and not conditioning on $\{\Gamma^\circ \geqslant X_1\}$) is the appropriate way to change the subprobability distribution $\mathbf{P}^\circ(C_1 \in \cdot, \Gamma^\circ \geqslant X_1)$ into the probability distribution of the i.i.d. cycles $\dots, \tilde{C}_{-1}, \tilde{C}_0$. Further, it turns out that these cycles should be independent of the future, $\theta_0(\tilde{Z}, \tilde{S}, \tilde{\Gamma})$. Finally, it turns out that an exponential biasing of the taboo time Γ° is the appropriate way to change the subprobability distribution $\mathbf{P}^\circ((Z^\circ, S^\circ, \Gamma^\circ) \in \cdot, \Gamma^\circ < X_1)$ into the probability distribution of $\theta_0(\tilde{Z}, \tilde{S}, \tilde{\Gamma})$.

10.4 Construction of $(\tilde{Z}, \tilde{S}, \tilde{\Gamma})$

We shall now turn to the construction of $(\tilde{Z}, \tilde{S}, \tilde{\Gamma})$ motivated intuitively above.

Theorem 10.3. (*a*) *Let* (Z, S, Γ) *be taboo regenerative, that is, let it satisfy* (9.1). *Suppose*

$$\begin{aligned} &\textit{there is an } \alpha > 0 \textit{ such that } \mathbf{E}^\circ[e^{\alpha X_1}\, 1_{\{\Gamma^\circ \geqslant X-1\}}] = 1 \\ &\textit{and } \mathbf{E}^\circ[e^{\alpha \Gamma^\circ}\, 1_{\{\Gamma^\circ < X_1\}}] < \infty. \end{aligned} \tag{10.3}$$

Define probability measures $\mathbf{P}_0^\circ, \mathbf{P}_1^\circ, \dots$ *on* $(\Omega, \mathcal{F})$ *by*

$$\mathbf{dP}_n^\circ := \frac{e^{\alpha\Gamma^\circ}\, 1_{\{S_n^\circ \leqslant \Gamma^\circ < S_{n+1}^\circ\}}}{\mathbf{E}^\circ[e^{\alpha\Gamma^\circ}\, 1_{\{\Gamma^\circ < X_1\}}]} \mathbf{dP}^\circ, \quad n \geqslant 0.$$

Then there exists a two-sided process $\tilde{Z} = (\tilde{Z}_s)_{s\in\mathbb{R}}$, *a nondecreasing two-sided sequence of* $(-\infty, \infty]$ *valued random times* $\tilde{S} = (\tilde{S}_k)_{-\infty}^{\infty}$ *satisfying, for each* $n \geqslant 1$,

$$-\infty \leftarrow \dots < \tilde{S}_{-2} < \tilde{S}_{-1} < 0 = \tilde{S}_0 < \dots < \tilde{S}_n \quad \text{on } \{\tilde{S}_n < \infty\},$$

and a finite nonnegative random time $\tilde{\Gamma}$ *such that*

$$\mathbf{P}(\theta_{\tilde{S}_{-n}}(\tilde{Z}, \tilde{S}, \tilde{\Gamma}) \in \cdot) = \mathbf{P}_n^\circ((Z^\circ, S^\circ, \Gamma^\circ) \in \cdot), \quad n \geqslant 0. \tag{10.4}$$

Moreover, the cycles of $(\tilde{Z}, \tilde{S})$ *up to time zero,*

$$\tilde{C}_k := (\tilde{Z}_{\tilde{S}_{k-1}+s})_{s\in[0,\tilde{X}_k)}, \quad k \leqslant 0, \qquad (\textit{here } \tilde{X}_k = \tilde{S}_k - \tilde{S}_{k-1})$$

are i.i.d. with distribution $\mathbf{P}^\circ_{\text{taboo}}(C_1 \in \cdot)$, *where* $\mathbf{P}^\circ_{\text{taboo}}$ *is the probability measure on* $(\Omega, \mathcal{F})$ *defined (as in Section 9.6) by*

$$\mathbf{dP}^\circ_{\text{taboo}} := e^{\alpha X_1}\, 1_{\{\Gamma^\circ \geqslant X_1\}}\, \mathbf{dP}^\circ.$$

Finally, these cycles are independent of $\theta_0(\tilde{Z}, \tilde{S}, \tilde{\Gamma})$, *which has the distribution* $\mathbf{P}_0^\circ((Z^\circ, S^\circ, \Gamma^\circ) \in \cdot)$.