2 Preliminaries – Stationarity

This section lays down the framework to be used in Sections 3 through 6 (and also with certain modifications in Sections 7 through 9) and then considers briefly the one-sided counterpart of the two-sided stationarity theory in Chapter 8.

2.1 The One-Sided Process and Points

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space supporting

$$Z = (Z_s)_{s \in [0,\infty)} \quad \text{and} \quad S = (S_k)_0^\infty,$$

where Z is a one-sided continuous-time stochastic process with a general state space $(E, \mathcal{E})$ and path space $(H, \mathcal{H})$ and S is a one-sided sequence of random times (points) satisfying

$$0 \leqslant S_0 < S_1 < \cdots \to \infty.$$

Regard S as a measurable mapping from $(\Omega, \mathcal{F})$ to the *sequence space* $(L, \mathcal{L})$, where

$$L = \{(s_k)_0^\infty \in [0,\infty)^{\{0,1,\dots\}} : s_0 < s_1 < \cdots \to \infty\}$$

and $\mathcal{L}$ are the Borel subsets of L, that is,

$$\mathcal{L} = L \cap B^{\{0,1,\dots\}}.$$

Thus the pair (Z, S) is a measurable mapping from $(\Omega, \mathcal{F})$ to $(H \times L, \mathcal{H} \otimes \mathcal{L})$. Let $\mathcal{H} \otimes \mathcal{L}^+$ denote the class of all measurable functions from $(H \times L, \mathcal{H} \otimes \mathcal{L})$ to $([0,\infty), \mathcal{B}[0,\infty))$.

We shall not assume any functional connection between Z and S. At one extreme, Z and S could be independent. At another extreme, S could be determined by Z: for instance, S could be the times when Z enters a given state or set. At a third extreme, Z could be determined by S, as is the case if Z is one of the following processes.

Let S_{-1} be a strictly negative random variable and put, for $t \in [0,\infty)$,

$$N_t = \inf\{n \geqslant 1 : S_n > t\} \qquad \text{number of points in } [0,t],$$
$$A_t = t - S_{N_t - 1} \qquad \text{age at time } t,$$
$$B_t = S_{N_t} - t \qquad \text{residual life at time } t,$$
$$D_t = X_{N_t} = A_t + B_t \qquad \text{total life at time } t,$$
$$U_t = A_t / D_t \qquad \text{relative age at time } t;$$

see Figures 8.1 and 9.1 in Chapter 2.

2.2 The One-Sided Shift – Shift-Measurability

For $t \in [0, \infty)$, let θ_t be the *shift-map* from H to H:

$$\theta_t z = (z_{t+s})_{s\in[0,\infty)}.$$

Let θ_t also denote the *joint shift-map* from $H \times L$ to $H \times L$:

$$\theta_t(z, (s_k)_0^\infty) = (\theta_t z, (s_{n_{t-}+k} - t)_0^\infty),$$

$$\text{where } n_{t-} = \inf\{n \geqslant 1 : s_n \geqslant t\}.$$

Note that θ_t is a *time* shift and shifts $(s_k)_0^\infty$ regarded as a sequence of *times*, that is, θ_t shifts $(s_k)_0^\infty$ by subtracting t from the times s_k and only shifts the index k of $(s_k)_0^\infty$ to observe the convention that the first time is indexed by 0.

In order to be able to shift at will without measurability complications, assume that Z is *shift-measurable*, that is, let the path set H be invariant under time-shifts and the mapping taking $(z, t) \in H \times [0, \infty)$ to $z_t \in E$ be $\mathcal{H} \otimes \mathcal{B}[0, \infty)/\mathcal{E}$ measurable. This is equivalent to the mapping taking $(z, t) \in H \times [0, \infty)$ to $\theta_t z \in H$ being $\mathcal{H} \otimes \mathcal{B}/\mathcal{H}$ measurable. Shift-measurability covers, for instance, processes with a Polish state space (in fact separable metric suffices) and right-continuous paths (left-hand continuity is not needed). See Section 2 of Chapter 4 for more details.

2.3 Cycles and Cycle-Lengths – Delay and Delay-Length

The random times S_n split Z into a *delay*

$$D = (Z_s)_{s\in[0,S_0)} \qquad \text{(see Figure 2.1)}$$

and a sequence of *cycles*

$$C_n = (Z_{S_{n-1}+s})_{s\in[0,X_n)}, \quad n \geqslant 1,$$

where X_n are the *cycle-lengths*

$$X_n = S_n - S_{n-1}, \quad n \geqslant 1.$$

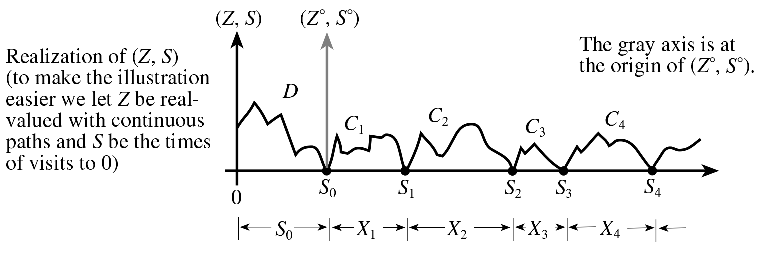

FIGURE 2.1. The points S split Z into a delay D and cycles C_n.

The delay D and the cycles C_n are stochastic processes vanishing at the random times S_0 and X_n, respectively. The easiest way to make sense of such processes as random elements is to think of them as entering an absorbing state Δ when vanishing, where Δ (the cemetery) is external to the state space; see Section 2.9 of Chapter 4 for technical details. The cycle-lengths $X_1, X_2, \ldots$ and the *delay-length* S_0 are all obtained by the same measurable mapping from their respective cycles $C_1, C_2, \ldots$ and delay D. They are simply the hitting times of the absorbing cemetery state Δ. The pair (Z, S) is a measurable mapping of the delay and cycles (string them together), and vice versa.

Say that (Z, S) is *zero-delayed* if $S_0 \equiv 0$. Define a zero-delayed pair by

$$(Z^\circ, S^\circ) := \theta_{S_0}(Z, S) \qquad \text{(see Figure 2.1)}.$$

Thus $S_0^\circ \equiv 0$ and $S_1^\circ \equiv X_1^\circ$, while for $n \geqslant 1, X_n^\circ \equiv X_n$ and $C_n^\circ \equiv C_n$.

2.4 Cycle-Stationarity – Stationarity

Call (Z, S) *cycle-stationary* if the cycles form a stationary sequence, that is, with $\stackrel{D}{=}$ denoting identity in distribution:

$$(C_{n+1}, C_{n+2}, \ldots) \stackrel{D}{=} (C_1, C_2, \ldots), \quad n \geqslant 0.$$

Cycle-stationarity is equivalent to

$$\theta_{S_n}(Z, S) \stackrel{D}{=} (Z^\circ, S^\circ), \quad n \geqslant 0,$$

since $(C_{n+1}, C_{n+2}, \ldots)$ and $\theta_{S_n}(Z, S)$ are measurable mappings of each other, and since these mappings do not depend on n. When (Z, S) is cycle-stationary, put

$$F(x) = \mathbf{P}(X_1 \leqslant x), \quad 0 \leqslant x < \infty,$$

that is, F is the common distribution function of the cycle lengths.

A pair (Z^*, S^*) is *stationary* if

$$\theta_t(Z^*, S^*) \stackrel{D}{=} (Z^*, S^*), \quad t \geqslant 0.$$

We now construct a stationary (Z^*, S^*) from a cycle-stationary (Z°, S°) when $\mathbf{E}[X_1] < \infty$. The proof is based on the same idea as in Section 9 of Chapter 2 and Section 4 of Chapter 8; namely, a stationary version should be obtained by length-biasing the first cycle of (Z°, S°) and then placing the time origin at random in that cycle.

Theorem 2.1. *Suppose (Z, S) is cycle-stationary with $\mathbf{E}[X_1] < \infty$. Let U be uniformly distributed on $[0, 1)$ and independent of (Z°, S°) and let $\mathbf{P}^*$ be the probability measure on $(\Omega, \mathcal{F})$ defined by*

$$\mathbf{dP}^* = \frac{X_1}{\mathbf{E}[X_1]}\mathbf{dP} \qquad (\textit{length-biasing}).$$