REMARK 2.1. The two stationary versions (Z^*, S^*) and $(\tilde{Z}, \tilde{S})$ of a cycle-stationary (Z°, S°) turn out to coincide in the regenerative case. This can be seen, for instance, by noting that there is a trivial successful distributional shift-coupling of a regenerative (Z°, S°) and the stationary version (Z^*, S^*) (see Theorem 3.1 below: the shift-coupling times are $S_0^\circ = 0$ and S_0^*). Since there is also successful distributional shift-coupling of a regenerative (Z°, S°) and the stationary version $(\tilde{Z}, \tilde{S})$, the two versions are both Cesaro total variation limits of (Z°, S°) and thus must have the same distribution.

REMARK 2.2. The reader may wonder why we state the Cesaro total variation and shift-coupling results only for Z° and $\tilde{Z}$ and not for (Z°, S°) and $(\tilde{Z}, \tilde{S})$ as in Chapter 8. This is just to be in accordance with the rest of this chapter. To simplify notation in this chapter we shall state many results for the process only and not for the joint process and points. This is no restriction because we can always embed S in the process by replacing Z by $(Z_s, A_s)_{s\in[0,\infty)}$. Then S is simply formed by the times when the age process $(A_s)_{s\in[0,\infty)}$ enters the state zero.

3 Classical Regeneration

In this section we shall consider processes regenerative in the sense commonly associated with that term. In order to distinguish this regeneration concept from the generalizations studied in Sections 4 through 10 we shall use the term *classical* regeneration.

3.1 Definition

Call a one-sided shift-measurable stochastic process Z *classical regenerative* with *regeneration times* S if

$$\theta_{S_n}(Z, S) \stackrel{D}{=} (Z^\circ, S^\circ), \quad n \geqslant 0, \tag{3.1}$$

and

$$\theta_{S_n}(Z, S) \text{ is independent of } ((Z_s)_{s\in[0,S_n)}, S_0, \dots, S_n), \ \ n \geqslant 0. \tag{3.2}$$

Call the pair (Z, S) *classical regenerative* if this holds.

This definition can be reformulated as follows: (Z, S) is classical regenerative if and only if

$$C_1, C_2, \dots \text{ are i.i.d. and independent of } D. \tag{3.3}$$

In order to establish this equivalence, first note that (3.3) can be reformulated as the following two claims

$$(C_{n+1}, C_{n+2}, \dots) \stackrel{D}{=} (C_1, C_2, \dots), \quad n \geqslant 0, \tag{3.4}$$

$$(C_{n+1}, C_{n+2}, \dots) \text{ is independent of } (D, C_1, \dots, C_n), \ \ n \geqslant 0; \tag{3.5}$$

then recall that (3.4) is equivalent to (3.1); and finally note that (3.5) is equivalent to (3.2), since $(C_{n+1}, C_{n+2}, \dots)$ and $\theta_{S_n}(Z, S)$ are measurable maps of each other and so are $(D, C_1, \dots, C_n)$ and $((Z_s)_{s \in [0, S_n)}, S_0, \dots, S_n)$.

It follows from (3.3) that if (Z, S) is classically regenerative, then the cycle-lengths $X_1, X_2, \dots$ are i.i.d. and independent of the delay-length S_0, that is, S is a *renewal process*. The cycle-lengths are also called *inter-regeneration times* and *recurrence times*. Let the nonnegative random variable S_{-1} be such that (S_{-1}, D) is independent of (Z°, S°).

3.2 Examples

Here are a few standard examples of processes that are regenerative in the classical sense.

Let S be a renewal process. Then the age process $(A_s)_{s \in [0,\infty)}$, the residual life process $(B_s)_{s \in [0,\infty)}$, the total life process $(D_s)_{s \in [0,\infty)}$, and the relative age process $(U_s)_{s \in [0,\infty)}$ are all classical regenerative with S as regeneration times. In fact, these processes viewed jointly as $(A_s, B_s, D_s, U_s)_{s \in [0,\infty)}$ form a four-dimensional classical regenerative process. Also, if Z is classical regenerative with regeneration times S, then so is $(Z_s, A_s, B_s, D_s, U_s)_{s \in [0,\infty)}$.

Let Z be an irreducible recurrent Markov chain (as in Chapter 2). Then Z is classical regenerative with regeneration times S formed by the successive entrances to a fixed reference state.

Let Z be a general state space shift-measurable Markov process (as in Chapter 6). Let A be a set of states such that Z enters A infinitely often and finitely many times in finite intervals, and satisfies the Markov property at the entrance times. If the transition probabilities are the same from all states in A, then Z is classical regenerative with regeneration times S formed by the successive entrances to A. Conversely, any classical regenerative (Z, S) can be embedded into a Markov process. For instance, the process with value $(Z_{t+s})_{s \in [0, B_t)}$ at time t is Markovian, and so is the process with value $(Z_{t+s})_{s \in [-A_t, 0]}$ at time t.

Regeneration is usually not a primary assumption in applications. Rather regeneration is the key property of many processes that come out of stochastic models, the property that makes the processes amenable to analysis. In particular, regenerative processes abound in queueing theory. As an example, let us consider the $GI/GI/1$ queueing model, namely the single-server queueing system where customers arrive to a service station at times forming a renewal process and line up to be served under the first-come-first-served discipline with i.i.d. service times that are independent of the arrival process (GI stands for *general independent*). Let α denote an inter-arrival time and β a services time. Let Q_t denote the queue length at time t and R_t the remaining service time of the customer being served at time t. If $\mathbf{E}[\beta] < \mathbf{E}[\alpha]$, then (according to the law of large numbers) the system will empty infinitely often and the bivariate process $(Q_s, R_s)_{s \in [0,\infty)}$ is classical regenerative with the times of arrivals to an empty system as regeneration