REMARK 3.4. Let S be a renewal process and S° its zero-delayed version. Consider this special case of (3.26): for all $h > 0$,

$$\|\mathbf{P}(B_{t+Uh} \in \cdot) - \mathbf{P}(B^\circ_{t+Uh}) \in \cdot)\| \to 0, \quad t \to \infty. \tag{3.34}$$

Theorem 3.4 has the following converse.

If S_0 is exponential and (3.34) holds, then

$$X_1 \text{ is nonlattice,}$$

since otherwise there is a $d > 0$ such that $\mathbf{P}(B^\circ_{nd+Ud/2} \in (d/2, d] + d\mathbb{Z}) = 1$ and $\mathbf{P}(B_{nd+Ud/2} \in (0, d/2] + dZ) \geqslant \mathbf{P}(d/2 < S_0 < d)$ for all n, implying $\|\mathbf{P}(B_{nd+Ud/2} \in \cdot) - \mathbf{P}(B^\circ_{nd+Ud/2} \in \cdot)\| \geqslant 2\mathbf{P}(d/2 < S_0 < d) > 0$ for all n, and thus contradicting (3.34).

4 Wide-Sense Regeneration – Harris Chains – *GI/GI/k*

It turns out that all the results from the previous section (except those on mixing and triviality) still hold if we allow the future after regeneration to depend on the past, as long as the future is independent of the past regeneration times. For lack of a better term we shall call this *wide-sense* regeneration. If the dependence lasts only over a time interval of length l, then the regeneration is *lag-l* (in this case the results on mixing and triviality hold). At the end of this section we show that this kind of regeneration occurs in so-called Harris chains and in the $GI/GI/k$ queueing system.

4.1 Definitions

Call a one-sided shift-measurable stochastic process Z *wide-sense regenerative* with *regeneration times* S if

$$\theta_{S_n}(Z, S) \stackrel{D}{=} (Z^\circ, S^\circ), \quad n \geqslant 0, \tag{4.1}$$

and

$$\theta_{S_n}(Z, S) \text{ is independent of } (S_0, \dots, S_n), \quad n \geqslant 0. \tag{4.2}$$

Call the pair (Z, S) *wide-sense regenerative* if this holds. If (Z, S) is wide-sense regenerative, then the cycles are in general not i.i.d. but S is still a renewal process. Let the nonnegative random variable S_{-1} be such that (S_{-1}, S_0) is independent of (Z°, S°).

With $l \geqslant 0$, call a wide-sense regenerative (Z, S) *lag-l regenerative* if (4.2) can be strengthened to: for $n \geqslant 0$,

$$\theta_{S_n}(Z, S) \text{ is independent of } ((Z_s)_{s \in [0,(S_n - l)^+]}, S_0, \dots, S_n); \tag{4.3}$$

and *lag-l+ regenerative* if (4.2) can only be strengthened to: for $n \geqslant 0$,

$$\theta_{S_n}(Z, S) \text{ is independent of } ((Z_s)_{s\in[0,(S_n-l)^+)}, S_0, \dots, S_n).$$

Thus lag-0+ regeneration is the same as classical regeneration, while lag-0 regeneration implies further that Z_{S_n} is nonrandom.

A pair (Z', S') is a *version* of a wide-sense regenerative (Z, S) if (Z', S') is also wide-sense regenerative and

$$\theta_{S'_0}(Z', S') \stackrel{D}{=} (Z^\circ, S^\circ). \tag{4.4}$$

A pair (Z', S') is a *version* of a lag-l regenerative (Z, S) if (Z', S') is also lag-l regenerative and (4.4) holds. In both cases (Z°, S°) is a zero-delayed version of (Z, S).

Note that if (Z, S) is classical regenerative, then (Z, S) is in particular wide-sense regenerative, and that if (Z', S') is a *wide-sense* version of (Z, S), then (Z', S') need not be a *classical* version of (Z, S), since the delay of (Z', S') need not be independent of the cycles.

4.2 Observations – Examples

Suppose (Z, S) is classical regenerative and f is a measurable mapping from $(H, \mathcal{H})$ into some measurable space. Then the process $(f(\theta_s Z))_{s\in[0,\infty)}$ is in general not classical regenerative. It is, however, wide-sense regenerative with regeneration times S. In particular, the *path process* $(\theta_s Z)_{s\in[0,\infty)}$ [which has state space $(H, \mathcal{H})$] is wide-sense regenerative with regeneration times S but certainly not classical regenerative (unless Z is a nonrandom constant).

In fact, the following conservation properties hold:

> If Z is classical regenerative with regeneration times S, then so is $(f(Z_s))_{s\in[0,\infty)}$ for all measurable mappings f from $(E, \mathcal{E})$ into some measurable space.
>
> If Z is wide-sense regenerative with regeneration times S, then so is $(f(\theta_s Z))_{s\in[0,\infty)}$ for all measurable mappings f from $(H, \mathcal{H})$ into some measurable space.

Note that a Markov process does not have these conservation properties:

> If Z is a Markov process, then in general $(f(\theta_s Z))_{s\in[0,\infty)}$ is not a Markov process and neither is $(f(Z_s))_{s\in[0,\infty)}$.

On the other hand, the following holds:

> For any stochastic process Z the path process $(\theta_s Z)_{s\in[0,\infty)}$ is always a Markov process.