Proof. Take $A \in \mathcal{E}_1$, let $q_\infty(A|\cdot)$ be a version of $\mathbf{P}(V \in A|W = \cdot)$, and put $B^+ = \{w \in E_2 : q_\infty(A|w) \geqslant q_t(A|w)\}$. Then

$$\begin{aligned}
&\mathbf{E}[1_{\{W \in B^+\}}|q_\infty(A|W) - q_t(A|W)|] \\
&\quad = \mathbf{P}(V \in A, W \in B^+) - \mathbf{P}(V_t \in A, W_t \in B^+) \\
&\qquad + \int_{B^+} q_t(A|w)\mathbf{P}(W_t \in dw) - \int_{B^+} q_t(A|w)\mathbf{P}(W \in dw).
\end{aligned}$$

The first difference on the right-hand side of this identity is dominated by $\frac{1}{2}\|\mathbf{P}((V,W) \in \cdot) - \mathbf{P}((V_t, W_t) \in \cdot)\|$, and the second difference is dominated by $\frac{1}{2}\|\mathbf{P}(W_t \in \cdot) - \mathbf{P}(W \in \cdot)\|$. Thus

$$\begin{aligned}
&\mathbf{E}[1_{\{W \in B^+\}}|q_\infty(A|W) - q_t(A|w)|] \\
&\quad \leqslant \|\mathbf{P}((V_t, W_t) \in \cdot) - \mathbf{P}((V,W) \in \cdot)\|.
\end{aligned}$$

With $B^- = \{w \in E_2 : q_\infty(A|w) < q_t(A|w)\}$ we obtain in the same way

$$\begin{aligned}
&\mathbf{E}[1_{\{W \in B^-\}}|q_\infty(A|W) - q_t(A|w)|] \\
&\quad \leqslant \|\mathbf{P}((V_t, W_t) \in \cdot) - \mathbf{P}((V,W) \in \cdot)\|.
\end{aligned}$$

Add these two inequalities and take the supremum in $A \in \mathcal{E}_1$ to get (5.23).

In particular, if $q_t(A|\cdot) = q(A|\cdot)$ for $t < \infty$, then it follows from (5.23) that $\mathbf{E}[|q(A|W) - \mathbf{P}(V \in A|W)|] = 0$, that is, $\mathbf{P}(V \in A|W) = q(A|W)$ a.s. as desired. □

6 Classical Coupling

In this section we shall use the classical coupling idea (Chapter 2) to construct a successful distributional exact coupling under the conditions of Theorem 5.3, that is, we shall complete the proof of Theorem 5.3.

Classical coupling in the context of time-inhomogeneous regenerative processes means simply using the time of first simultaneous regeneration of two independent versions of the process as a distributional coupling time. This procedure is successful under the lattice conditions of Theorem 5.3 and can be modified to be successful under the nonlattice conditions.

Throughout this section and the next we shall write $\mathbf{P}_s$ to indicate that $S_0 = s$ with probability one.

6.1 Classical Coupling – The Lattice Case

We start by showing that the classical coupling is successful under the lattice condition of Theorem 5.3.

Theorem 6.1. *Let S and S' be independent nonnegative Markov sequences increasing strictly to infinity with common time recurrence distributions F_s,*