PROOF. For $B \in \bigcup_{k=0}^{\infty} \mathcal{E}_0 \otimes \cdots \otimes \mathcal{E}_k$ we have

$$\begin{aligned}
&\mathbf{P}(A \cap \{(Y_0, \ldots, Y_J) \in B\}) \\
&\quad = \sum_{n=0}^{\infty} \mathbf{P}(A \cap \{(Y_0, \ldots, Y_n) \in B, J = n\}) \\
&\quad = \sum_{n=0}^{\infty} \mathbf{E}[1_{\{(Y_0,\ldots,Y_n)\in B\}} 1_{\{J=n\}} \mathbf{P}(A|Y_0, \ldots, Y_n)] \\
&\quad = \mathbf{E}[1_{\{(Y_0,\ldots,Y_J)\in B\}} \sum_{n=0}^{\infty} 1_{\{J=n\}} \mathbf{P}(A|Y_0, \ldots, Y_n)].
\end{aligned}$$

Therefore $\sum_0^\infty 1_{\{J=n\}} \mathbf{P}(A|Y_0, \ldots, Y_n)$ is a version of $\mathbf{P}(A|Y_0, \ldots, Y_J)$ as desired. □

7 The Coupling Time – Rates and Uniformity

In this section we shall take a closer look at the coupling time T constructed in the last section in order to obtain results on rates of convergence and uniform convergence along the lines of Section 6 in Chapter 4. After establishing a useful lemma we show that T can be stochastically dominated by a manageable random variable $\bar{T}$. This yields uniform total variation convergence over a class of processes (Theorem 7.1). We then establish finite moment results for $\bar{T}$, which yields rate results for the uniform convergence (Theorem 7.2). Finally, we establish sharper moment results for T itself, which yields improved (but not uniform) rate results (Theorem 7.3). At the end of the section we consider some consequences of this for classical and wide-sense regenerative processes, and improve Blackwell's renewal theorem in the spread-out case.

Throughout this section we assume that the conditions of Theorem 5.3 hold and write $\mathbf{P}_s$ to indicate that $S_0 = s$ with probability one.

7.1 Preliminaries

In the nonlattice case let $(\tau_k, \tau'_k, I_k)_0^\infty$, $(\beta_k, \beta'_k)_1^\infty$, and M be as in the proof of Theorem 6.2 above. In order to treat the lattice case and the nonlattice case simultaneously we shall not base our argument in the lattice case on the proof of Theorem 6.1 but rather define $(\tau_k, \tau'_k, I_k)_0^\infty$, $(\beta_k, \beta'_k)_1^\infty$, and M as in the proof of Theorem 6.2, replacing t_0 and p from Lemma 6.3 by $t_0 := n_0 d$ and p from Lemma 6.1 and with $c = 0$ and μ the distribution having mass 1 at zero, $\mu(\{0\}) = 1$. The proof of Theorem 6.2 works in the lattice case after this modification. Thus in both cases there are distributional coupling times T and T' for Z and Z' such that

$$T \stackrel{D}{=} \tau_M. \tag{7.1}$$