8 Asymptotics From-the-Past

How are things now (and from now on) if they started long ago?

The traditional probabilistic way to answer this loosely formulated question is to start a stochastic process at time 0, consider its distribution in a time interval $[t, \infty)$, and check whether it stabilizes as $t \to \infty$ (asymptotic stationarity); see Figure 8.1.

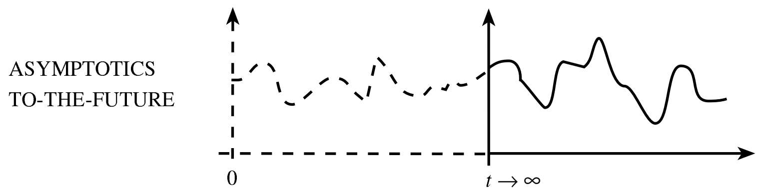

FIGURE 8.1. Realization of a process starting at time 0.

This we have done repeatedly up to now, obtaining asymptotic stationarity for Markov chains in Chapter 2 and for classical regenerative and wide-sense regenerative processes in this chapter. This approach did not work, however, for (truly) time-inhomogeneous regenerative processes: according to Theorem 5.5, asymptotic stationarity forces a time-inhomogeneous regenerative process to be time-*homogeneous* in the long run.

In this section we shall reverse this taking limits *to-the-future* approach as follows. We start a stochastic process at an arbitrary time r, consider its distribution in a *fixed* time interval $[t, \infty)$, and check whether it stabilizes as the *starting time* r goes *backward* to $-\infty$, $r \downarrow -\infty$; see Figure 8.2.

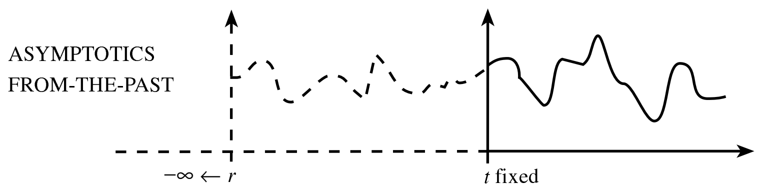

FIGURE 8.2. Realization of a process starting at time r.

As an answer to the above question, taking limits *from-the-past* in this way is even more natural than taking limits to-the-future. Of course, for time-homogeneous processes the two approaches are equivalent. The point is that unlike taking limits to-the-future, this taking limits from-the-past approach also works for time-inhomogeneous processes and thus widely extends the class of processes admitting a limit law.

In this section we establish that there is a limit from-the-past of time-inhomogeneous regenerative processes (wide-sense or not) satisfying the conditions from Theorem 5.3. The proof is based on the stochastic domination result in Theorem 7.1. At the end of the section we extend this result to processes that are time-inhomogeneous regenerative only up-to-time-zero.

8.1 Preliminaries

In order to take limits from-the-past we must consider processes with time set $[r,\infty)$. So fix an arbitrary $r \in (-\infty, 0]$ and add the following to the framework from Section 2. Let

$$Z^{(r)} = (Z_s^{(r)})_{s\in[r,\infty)}$$

be a one-sided stochastic process with time set $[r,\infty)$, state space $(E,\mathcal{E})$, and path set $H^{(r)}$ obtained from the internally shift-invariant subset H of $E^{[0,\infty)}$ by

$$H^{(r)} := \{(z_{s-r})_{s\in[r,\infty)} : (z_s)_{s\in[0,\infty)} \in H\}.$$

Let $\mathcal{H}^{(r)}$ be the trace of $H^{(r)}$ on $\mathcal{E}^{[r,\infty)}$. For $t \in [r,\infty)$, define the shift map θ_t on $H^{(r)}$ to be the map taking $z = (z_s)_{s\in[r,\infty)} \in H^{(r)}$ to

$$\theta_t z := (z_{t+s})_{s\in[0,\infty)} \in H.$$

Note that the process $\theta_r Z^{(r)}$ has time set $[0,\infty)$ and that the process $Z^{(r)}$ is shift-measurable if and only if $\theta_r Z^{(r)}$ is shift-measurable.

Let $S^{(r)} = (S_k^{(r)})_0^\infty$ be a one-sided sequence of random times satisfying

$$r \leqslant S_0^{(r)} < S_1^{(r)} < \cdots \to \infty.$$

Regard $S^{(r)}$ as a measurable mapping from $(\Omega,\mathcal{F})$ to the sequence space $(L^{(r)},\mathcal{L}^{(r)})$, where

$$L^{(r)} = \{(s_k)_0^\infty \in [r,\infty)^{\{0,1,\cdots\}} : s_0 < s_1 < \cdots \to \infty\} = r + L,$$

$$\mathcal{L}^{(r)} = L^{(r)} \cap \mathcal{B}^{\{0,1,\dots\}} = \text{ the Borel subsets of } L^{(r)}.$$

For $t \in [r,\infty)$, define the joint shift-map θ_t on $H^{(r)} \times L^{(r)}$ to be the map taking $(z,(s_k)_0^\infty) \in H^{(r)} \times L^{(r)}$ to

$$\theta_t(z,(s_k)_0^\infty) := (\theta_t z, (s_{n_{t-}+k} - t)_0^\infty) \in H \times L,$$

where $n_{t-} = \inf\{n \geqslant 1 : s_n \geqslant t\}$.

For $t \in \mathbb{R}$, define the shift-map θ_t on $E^{\mathbb{R}}$ to be the map taking $z = (z_s)_{s\in\mathbb{R}} \in E^{\mathbb{R}}$ to

$$\theta_t z := (z_{t+s})_{s\in[0,\infty)} \in E^{[0,\infty)}$$

and note that although $z = (z_s)_{s\in\mathbb{R}} \in E^{\mathbb{R}}$ is two-sided, the shift θ_t is one-sided.