define $Z^{(*,t)}$ by $\theta_t Z^{(*,t)} := \theta_t Z^{(*,-c)}$ to obtain (8.27a) and (8.27b) from the observation that when $t \in (-c, \infty)$, the left-hand side, $\theta_t Z^{(r)}$, is obtained from $\theta_{-c} Z^{(r)}$ by the same shift as the right-hand side, $\theta_t Z^{(*,t)}$, from $\theta_{-c} Z^{(*,-c)}$ [see (3.2) of Lemma 3.1 in Chapter 6].

In order to obtain the regeneration claim, proceed as in the first part of the proof of Theorem 8.2 to obtain the existence of a sequence of random times $S^{(*,t)}$ such that for $n \geqslant 0$ and $A \in \mathcal{H} \otimes \mathcal{L}_\infty$, the following holds a.s. on $\{S_n^{(*,t)} \leqslant 0\}$:

$$\mathbf{P}\big(\theta_{S_n^{(*,t)}}(Z^{(*,t)}, S^{(*,t)}) \in A | S_0^{(*,t)}, \ldots, S_n^{(*,t)}\big) = p(A | S_n^{(*,t)}).$$

In order to obtain the distributional coupling claim, apply Theorem 8.2 to the family $(W^{(r)}, R^{(r)})$, $r \in (-\infty, 0]$, to obtain, for each $t \in (-\infty, -c]$, finite random times $\tau^{(t)}$ and $\tau^{(*,t)}$ such that

$$\big(\theta_{\tau^{(t)}} W^{(t)}, \tau^{(t)}\big) \stackrel{D}{=} \big(\theta_{\tau^{(*,t)}} W^{(*,t)}, \tau^{(*,t)}\big) \quad \text{and} \quad \tau^{(t)} \stackrel{D}{\leqslant} \tilde{T}. \tag{8.31}$$

Define

$$T^{(t)} := \tau^{(t)} \text{ if } \tau^{(t)} \leqslant -t-c \qquad \text{and} \quad T^{(t)} := \infty \text{ if } \tau^{(t)} > -t-c,$$
$$T^{(*,t)} := \tau^{(*,t)} \text{ if } \tau^{(*,t)} \leqslant -t-c \quad \text{and} \quad T^{(*,t)} := \infty \text{ if } \tau^{(*,t)} > -t-c,$$

and recall that $W_t^{(t)} = \theta_t Z^{(t)}$ and $W_t^{(*,t)} = \theta_t Z^{(*,t)}$ for $t \in (-\infty, -c]$ to obtain from (8.31) that (8.28a) and (8.28b) hold. The nondistributional coupling claim now follows from Theorem 3.2 in Chapter 4.

If $(E, \mathcal{E})$ is Polish, repeat the proof of Theorem 8.3 to obtain from (8.27a) and (8.27b) that (8.28a) and (8.28b) hold for a two-sided Z^*, and that right continuity and left-hand limits of the paths transfer to Z^*. The nondistributional coupling claim follows from the fact that $(E, \mathcal{E})$ is Polish and the paths right-continuous.

Theorem 8.4 yields the moment results for $\tilde{T}$ and the rate and uniformity results for the convergence at (8.30a) and (8.30b). This yields the rate and uniformity results for the convergence at (8.27a) and (8.27b) [since the left- and right-hand sides at (8.27a) and (8.27b) are measurable mappings of the left- and right-hand sides at (8.30a) and (8.30b)]. The rate and uniformity results for the convergence at (8.28a) and (8.28b) follows immediately from this. □

9 Taboo Regeneration

Suppose we are studying a fish population that has lived a long time in an isolated lake. This fish population will eventually become extinct, but suppose it is still there at the time of observation. Then it is not appropriate to use asymptotic stationarity to motivate a stationary process as a model for the present state of the population. We should rather consider the asymptotic behaviour of the population under an extinction *taboo*, that is, conditionally on the observed fact that the population is still nonextinct at the time of observation. We should look for a *taboo limit*.

In this section we shall introduce *taboo regenerative* processes, processes that regenerate as long as some specific event (like extinction) has not occurred. This is the generalization of regeneration appropriate for obtaining a taboo limit. A key ingredient in our analyses will be the fact (Theorem 9.1 below) that the taboo conditioning turns taboo regeneration into time-inhomogeneous regeneration up-to-time-zero (up to the observation time). Therefore the asymptotics from-the-past in the previous section apply to yield a taboo limit (Theorem 9.4 below).

9.1 Preliminaries

For taboo purposes we need to modify the framework in Section 2 by allowing the sequence of times S to be terminating (to be absorbed at infinity). Let $S = (S_k)_0^\infty$ be a nondecreasing sequence of random times that is strictly increasing as long as it is finite, that is, for each $n \geqslant 0$,

$$0 \leqslant S_0 < S_1 < \cdots < S_n \qquad \text{on } \{S_n < \infty\}.$$

Regard S as a measurable mapping from $(\Omega, \mathcal{F})$ to the *sequence space* $(L_\infty, \mathcal{L}_\infty)$, where (with $s_{-1} = -\infty$)

$$L_\infty = \{(s_k)_0^\infty \in [0,\infty]^{\{0,1,\ldots\}} : s_{k-1} < s_k < \infty \text{ or } s_k = s_{k+1} = \infty\},$$

$$\mathcal{L}_\infty = L_\infty \cap \mathcal{B}[0,\infty]^{\{0,1,\ldots\}} \qquad \text{(the Borel subsets of } L_\infty).$$

Let $Z = (Z_s)_{s\in[0,\infty)}$ be as in Section 2 and let Γ be a finite nonnegative random time. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be the probability space supporting (Z, S, Γ) and assume that

$$\mathbf{P}(\Gamma \geqslant S_n) > 0, \qquad n \geqslant 0.$$

The triple (Z, S, Γ) is a measurable mapping from the measurable space $(\Omega, \mathcal{F})$ to $(H \times L_\infty \times [0,\infty), \mathcal{H} \otimes \mathcal{L}_\infty \otimes \mathcal{B}[0,\infty))$. As in Section 2, for $t \in [0,\infty)$, let θ_t be the shift-map from H to H

$$\theta_t z := (z_{t+s})_{s\in[0,\infty)}$$

and also the joint shift-map from $H \times L_\infty$ to $H \times L_\infty$:

$$\theta_t(z, (s_k)_0^\infty) := (\theta_t z, (s_{n_{t-}+k} - t)_0^\infty),$$

$$\text{where } n_{t-} = \inf\{n \geqslant 1 : s_n \geqslant t\}.$$